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15UEC904
LINEAR CONTROL ENGINEERING

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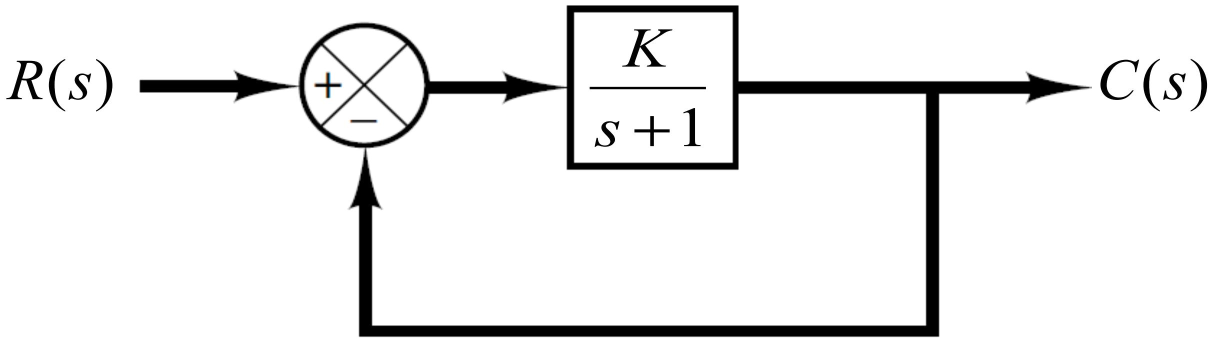
Root Locus: Introduction

- Consider a unity feedback control system shown below.



- The open loop transfer function G(s) of the system is

$$G(s) = \frac{K}{s+1}$$

- And the closed transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s+1+K}$$

Introduction

- The open loop stability does not depend upon gain K.

$$G(s) = \frac{K}{s+1}$$

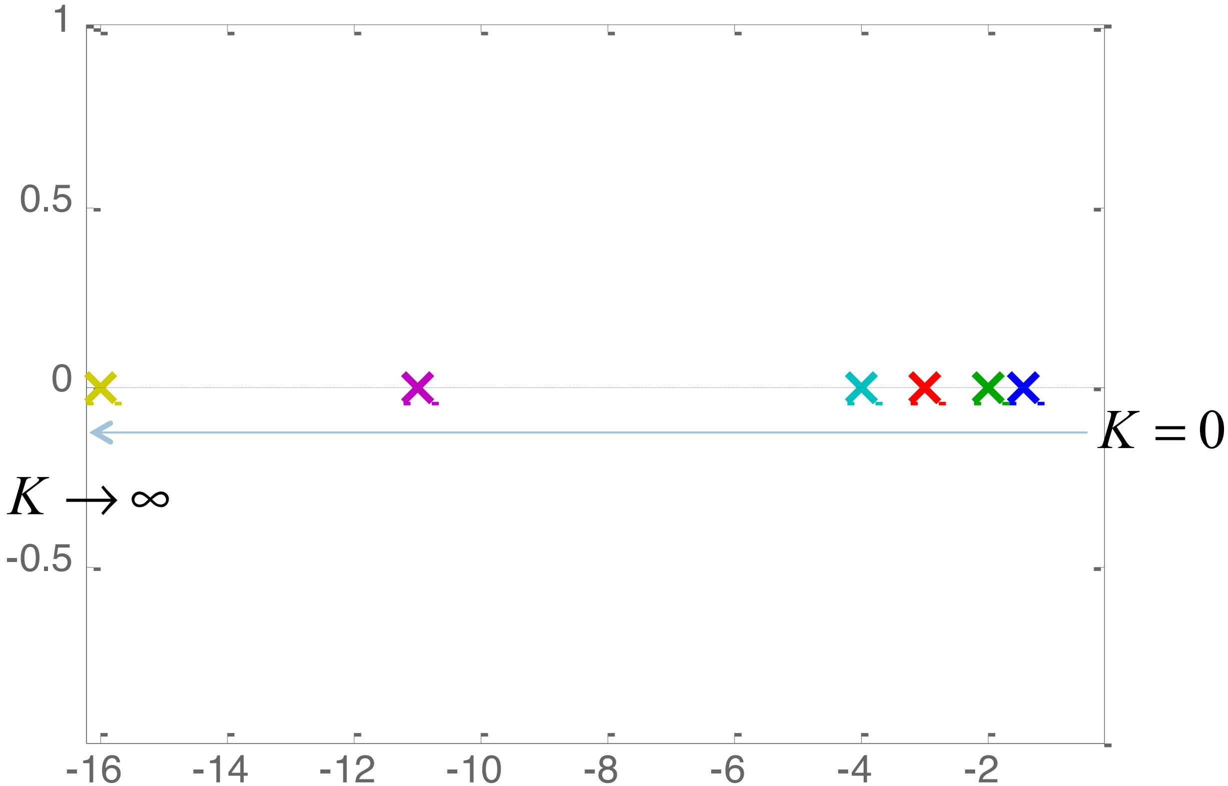
- Whereas, the location of closed loop poles vary with the variation in gain.

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

- Location of closed loop Pole for different values of K (remember K>0).

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16



What is Root Locus?

- The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter varies from zero to infinity.

How to Sketch root locus?

- One way is to compute the roots of the characteristic equation for all possible values of K.

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16

How to Sketch root locus?

- Computing the roots for all values of K might be tedious for higher order systems.

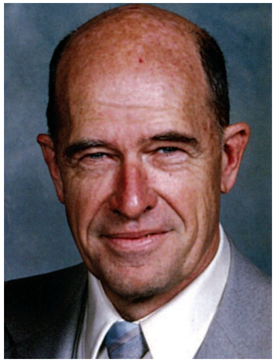
K	Pole
0.5	?
1	?
2	?
3	?
5	?
10	?
15	?

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+10)(s+20)+K}$$

Construction of Root Loci

- Finding the roots of the characteristic equation of degree higher than 3 is laborious and will need computer solution.

- A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering.



- This method, called the *root-locus method,* is one in which the roots of the characteristic equation are plotted for all values of a system parameter.

Construction of Root Loci

- The roots corresponding to a particular value of this parameter can then be located on the resulting graph.

- Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used.

- By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros.

PROCEDURE FOR CONSTRUCTING ROOT LOCUS

Step 1 : Locate the poles and zeros of G(s)H(s) on the s-plane. The root locus branch starts from open loop poles and terminates at zeros.

Step 2 : Determine the root locus on real axis.

Step 3 : Determine the asymptotes of root locus branches and meeting point of asymptotes with real axis.

Step 4 : Find the breakaway and breakin points.

Step 5 : If there is a complex pole then determine the angle of departure from the complex pole. If there is a complex zero then determine the angle of arrival at the complex zero.

Step 6 : Find the points where the root loci may cross the imaginary axis.

Step 7 : Take a series of test points in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy angle criterion. Sketch the root locus by joining the test points by smooth curve.

Step 8 : The value of gain K at any point on the locus can be determined from magnitude condition.

The value of K at a point $s = s_a$, is given by,

$$K = \frac{\text{product of length of vectors from poles to the point, } s = s_a}{\text{product of length of vectors from finite zeros to the point, } s = s_a}$$

Example

A unity feedback control system has an open loop transfer function, $G(s) = \dfrac{K}{s(s^2+4s+13)}$. Sketch the root locus.

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s^2+4s+13)=0$

The roots of the quadratic are, $s = \dfrac{-4 \pm \sqrt{4^2 - 4 \times 13}}{2} = -2 \pm j3$

$\therefore$ The poles are lying at $s = 0, -2+j3$ and $-2-j3$.

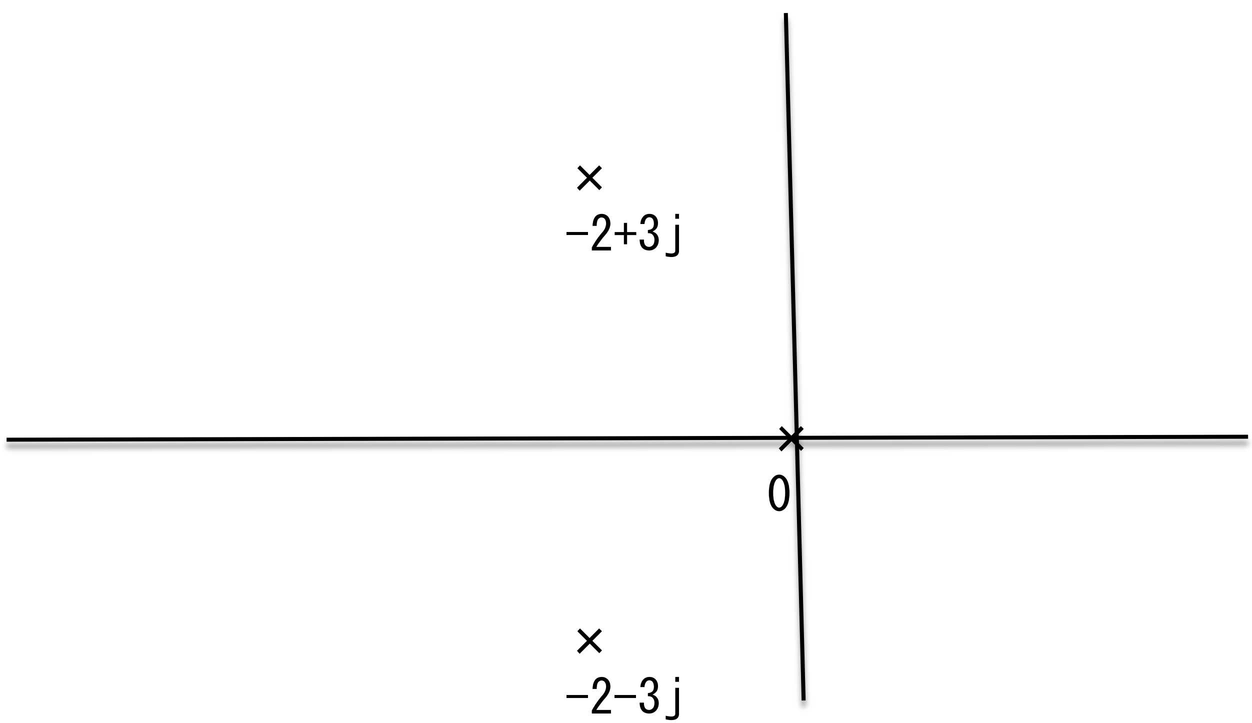
Let us denote the poles as P_1, P_2, and P_3.

Here, $P_1 = 0$, $P_2 = -2+j3$ and $P_3 = -2-j3$.

Step 2 : Root locus on real axis

In order to determine the part of root locus on real axis, take a test point on real axis. If the total number of poles and zeros on the real axis to the right of this test point is odd number, then the test point lies on the root locus. If it is even then the test point does not lie on the root locus.

There is only one pole on real axis at the origin. Hence if we choose any test point on the negative real axis then to the right of that point the total number of real poles and zeros is one, which is an odd number. Hence the entire negative real axis will be part of root locus. The root locus on real axis is shown as a bold line in fig .



Step 3 : To find angles of asymptotes and centroid

Since there are 3 poles, the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

$$\text{Angles of asymptotes} = \frac{\pm 180^{0}\ (2q+1)}{n-m} \quad ; \quad q = 0, 1, \ldots\ldots\ldots n-m$$

Here n = 3, and m = 0. $\quad \therefore q = 0, 1, 2, 3$.

n – no. of poles.
m – no. of zeros.

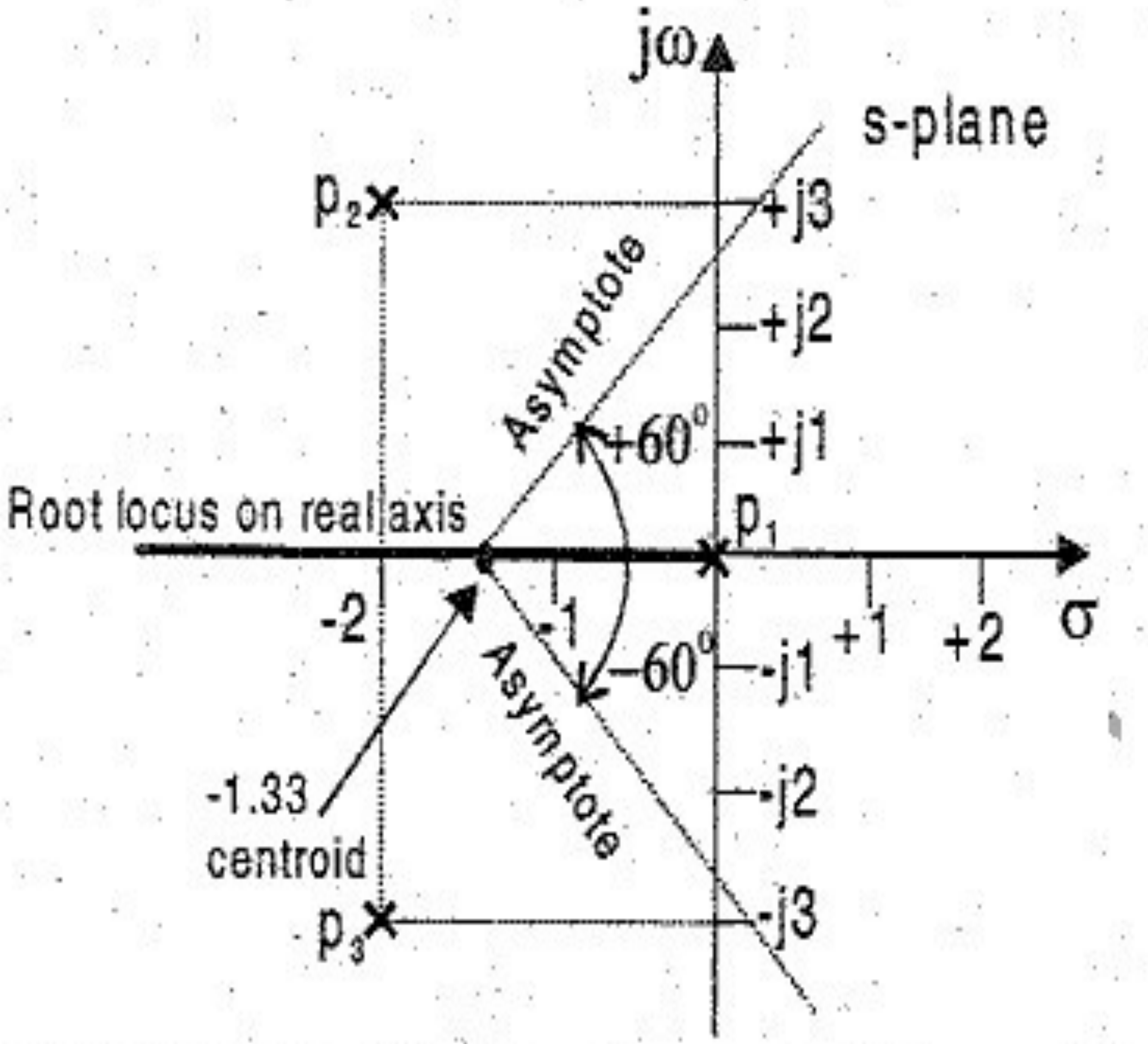
When $q = 0$, $\quad \text{Angles} = \pm \frac{180^{\circ}}{3} = \pm 60^{\circ}$

When $q = 1$, $\quad \text{Angles} = \pm \frac{180^{\circ} \times 3}{3} = \pm 180^{\circ}$

When $q = 2$, $\quad \text{Angles} = \pm \frac{180^{\circ} \times 5}{3} = \pm 300^{\circ} = \mp 60^{\circ}$

When $q = 3$, $\quad \text{Angles} = \pm \frac{180^{\circ} \times 7}{3} = \pm 420^{\circ} = \pm 60^{\circ}$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2+j3-2-j3-0}{3} = \frac{-4}{3} = -1.33$$


jω
s-plane
p2
+j3
+j2
+j1
Asymptote
+60°
Root locus on real axis
p1
-2
-1
+1
+2
σ
-60°
-j1
Asymptote
-1.33
centroid
-j2
-j3
p3

Step 4 : To find the breakaway and breakin points

The closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\dfrac{K}{s(s^2+4s+13)}}{1+\dfrac{K}{s(s^2+4s+13)}} = \frac{K}{s(s^2+4s+13)+K}$$

The characteristic equation is, $s(s^2+4s+13)+K=0$

$$\therefore \quad s^3+4s^2+13s+K=0 \quad \Rightarrow \quad K=-s^3-4s^2-13s$$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds} = -(3s^2+8s+13)$$

Put $\dfrac{dK}{ds}=0$

$$\therefore \quad -(3s^2+8s+13)=0 \quad \Rightarrow \quad (3s^2+8s+13)=0$$

$$\therefore s = \frac{-8 \pm \sqrt{8^2-4\times 13\times 3}}{2\times 3} = -1.33 \pm j1.6$$

Check for K : When, $s = -1.33 + j1.6$, the value of K is given by,

$$K = -(s^3 + 4s^2 + 13s) = -[(-1.33 + j1.6)^3 + 4(-1.33 + j1.6)^2 + 13(-1.33 + j1.6)]$$

$$\neq \text{positive and real.}$$

Also it can be shown that when $s = -1.33 - j1.6$ the value of K is not equal to real and positive.

Since the values of K for, $s = -1.33 \pm j1.6$, are not real and positive, these points are not an actual breakaway or breakin points. The root locus has neither breakaway nor breakin point.

Step 5 : To find the angle of departure

Let us consider the complex pole p_2 shown in fig 4.22.2. Draw vectors from all other poles to the pole p_2 as shown in fig 4.22.2. Let the angles of these vectors be θ_1 and θ_2.

Here, $\theta_1 = 180° - \tan^{-1}(3/2) = 123.7°$; $\theta_2 = 90°$

Angle of departure from the complex pole $p_2 = 180° - (\theta_1 + \theta_2)$

$$= 180° - (123.7° + 90°)$$

$$= -33.7°$$

The angle of departure at complex pole p_3 is negative of the angle of departure at complex pole A.

$\therefore$ Angle of departure at pole $p_3 = +33.7°$

Mark the angles of departure at complex poles using protractor.

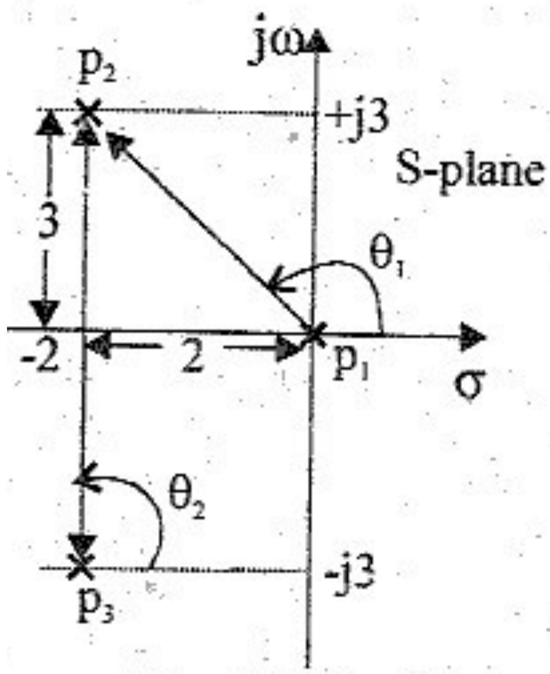


Fig 4.22.2

Step 6 : To find the crossing point on imaginary axis

The characteristic equation is given by,

$$s^3 + 4s^2 + 13s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 4(j\omega)^2 + 13(j\omega) + K = 0 \quad \Rightarrow -j\omega^3 - 4\omega^2 + 13j\omega + K = 0$$

On equating imaginary part to zero, we get,

$$-\omega^3 + 13\omega = 0$$

$$-\omega^3 = -13\omega$$

$$\omega^2 = 13 \quad \Rightarrow \quad \omega = \pm\sqrt{13} = \pm 3.6$$

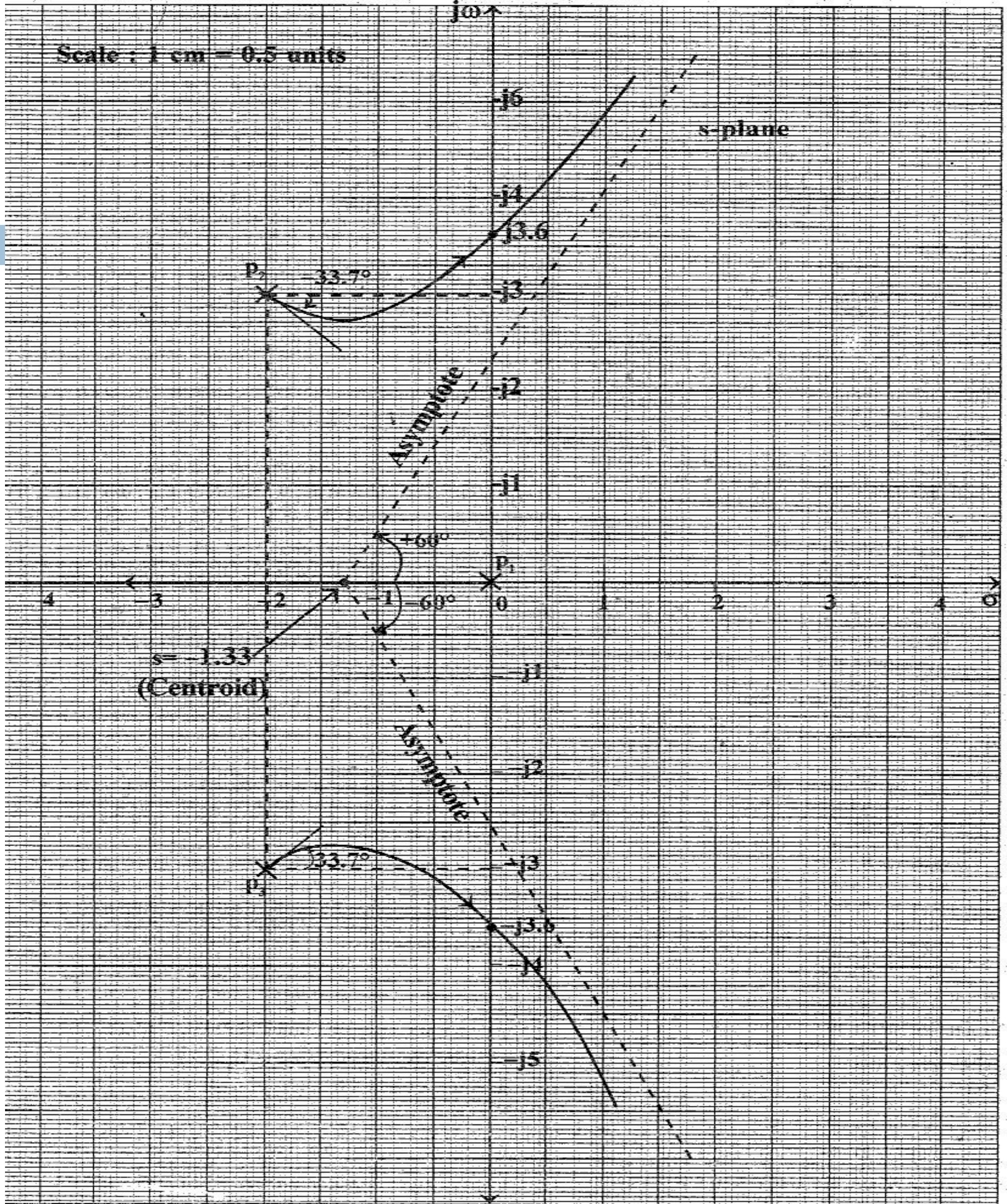
On equating real part to zero, we get,

$$-4\omega^2 + K = 0$$

$$K = 4\omega^2$$

$$= 4 \times 13 = 52$$

The crossing point of root locus is $\pm$ j3.6. The value of K at this crossing point is K = 52. (This is the limiting value of K for the stability of the system).



Scale : 1 cm = 0.5 units
jω
s-plane
j5
j4
j3.6
j3
j2
j1
−j1
−j2
−j3
−j3.6
−j4
−j5
p_1
p_2
p_3
−33.7°
33.7°
+60°
−60°
Asymptote
Asymptote
s= −1.33
(Centroid)
σ

EXAMPLE

Sketch the root locus of the system whose open loop transfer function is, $G(s) = \frac{K}{s\,(s+2)\,(s+4)}$. Find the value of K so that the damping ratio of the closed loop system is 0.5.

Step 1 : To locate poles and zeros

The poles of open loop transfer function are the roots of the equation, $s(s+2)\,(s+4) = 0$.

$\therefore$ The poles are lying at, $s = 0, -2, -4$.

Let us denote the poles as p_1, p_2, and p_3.

Here, $p_1 = 0$, $p_2 = -2$, $p_3 = -4$.

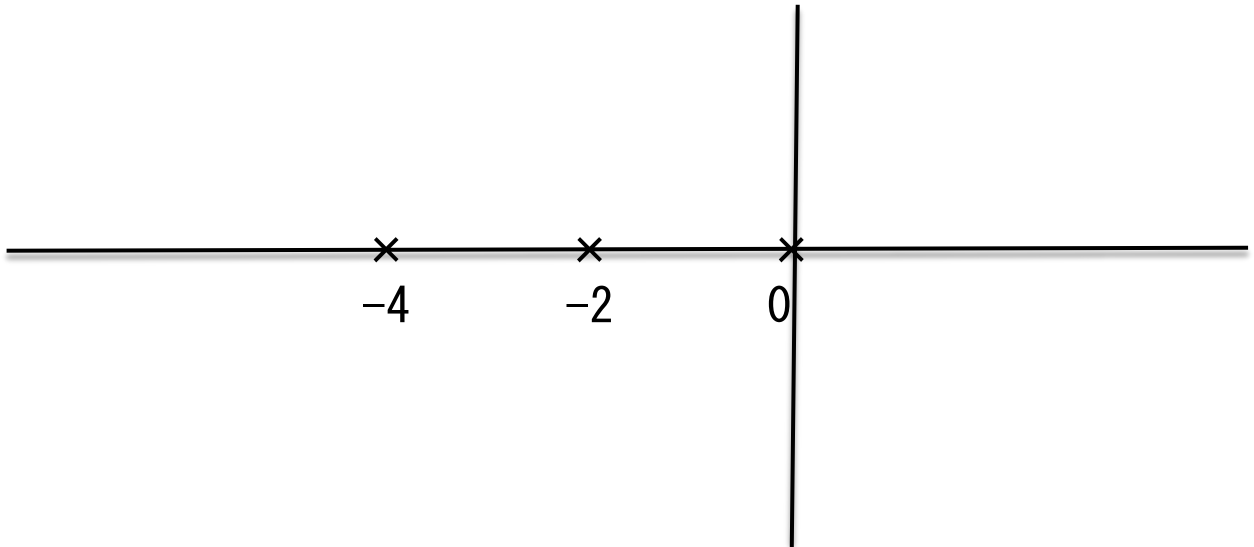
Step 2 : To find the root locus on real axis

There are three poles on the real axis.

Choose a test point on real axis between $s = 0$ and $s = -2$. To the right of this point the total number of real poles and zeros is one, which is an odd number. Hence the real axis between $s = 0$ and $s = -2$ will be a part of root locus.

Choose a test point on real axis between $s = -2$ and $s = -4$. To the right of this point, the total number of real poles and zeros is two which is an even number. Hence the real axis between $s = -2$ and $s = -4$ will not be a part of root locus.

Choose a test point on real axis to the left of $s = -4$. To the right of this point, the total number of real poles and zeros is three, which is an odd number. Hence the entire negative real axis from $s = -4$ to $-\infty$ will be a part of root locus.



Step 3 : To find asymptotes and centroid

Since there are three poles the number of root locus branches are three. There is no finite zero. Hence all the three root locus branches ends at zeros at infinity. The number of asymptotes required are three.

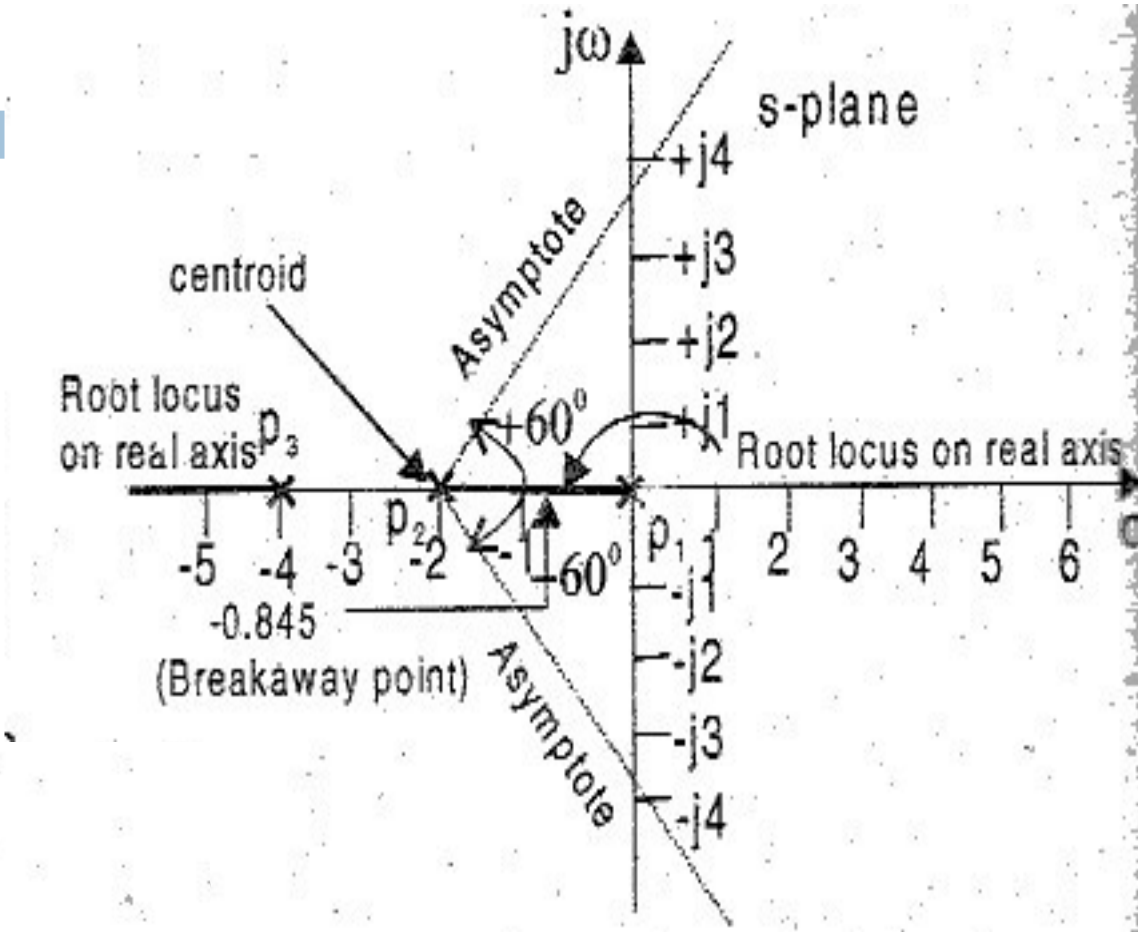
$$\text{Angles of asymptotes} = \frac{\pm 180^\circ (2q+1)}{n-m} \quad ; \quad q = 0, 1, 2, \ldots\ldots n-m.$$

Here, $n = 3$ and $m = 0$. $\therefore q = 0, 1, 2, 3$.

$$\text{When } q = 0, \quad \text{Angles} = \pm\frac{180^\circ}{3} = \pm 60^\circ$$

$$\text{When } q = 1, \quad \text{Angles} = \pm\frac{180^\circ \times 3}{3} = \pm 180^\circ$$

$$\text{Centroid} = \frac{\text{Sum of poles} - \text{Sum of zeros}}{n-m} = \frac{0-2-4-0}{3} = -2$$



jω
s-plane
+j4
+j3
+j2
+j1
-j1
-j2
-j3
-j4
Asymptote
Asymptote
centroid
Root locus
on real axis
Root locus on real axis
p3
p2
p1
+60°
-60°
-5
-4
-3
-2
-1
1
2
3
4
5
6
-0.845
(Breakaway point)

Step 4 : To find the breakaway and breakin points

The closed loop transfer function $\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{\frac{K}{s(s+2)(s+4)}}{1+\frac{K}{s(s+2)(s+4)}}=\frac{K}{s(s+2)(s+4)+K}$

The characteristic equation is given by,

$$s(s+2)(s+4)+K=0 \quad \Rightarrow \quad s(s^2+6s+8)+K=0 \quad \Rightarrow \quad s^3+6s^2+8s+K=0$$

$$\therefore K=-s^3-6s^2-8s$$

On differentiating the equation of K with respect to s we get,

$$\frac{dK}{ds}=-(3s^2+12s+8)$$

Put $\frac{dK}{ds}=0$

$$\therefore -(3s^2+12s+8)=0 \quad \Rightarrow \quad (3s^2+12s+8)=0$$

$$s=\frac{-12\pm\sqrt{12^2-4\times 3\times 8}}{2\times 3}=-0.845 \quad \text{or} \quad -3.154$$

Check for K : When s = –0.845, the value of K is given by,

$$K = -[(-0.845)^3 + 6(-0.845)^2 + 8(-0.845)] = 3.08$$

Since K, is positive and real for, s = –0.845, this point is actual breakaway point.

When s = –3.154, the value of K is given by,

$$K = -[(-3.154)^3 + 6(-3.154)^2 + 8(-3.154)] = -3.08$$

Since K, is negative for, s = –3.154, this is not a actual breakaway point.

Step 5 : To find angle of departure

Since there are no complex pole or zero, we need not find angle of departure or arrival.

Step 6 : To find the crossing point of imaginary axis

The characteristic equation is given by,

$$s^3 + 6s^2 + 8s + K = 0$$

Put $s = j\omega$

$$(j\omega)^3 + 6(j\omega)^2 + 8(j\omega) + K = 0$$

$$-j\omega^3 - 6\omega^2 + j8\omega + K = 0$$

Equating imaginary part to zero

$$-j\omega^3 + j8\omega = 0$$

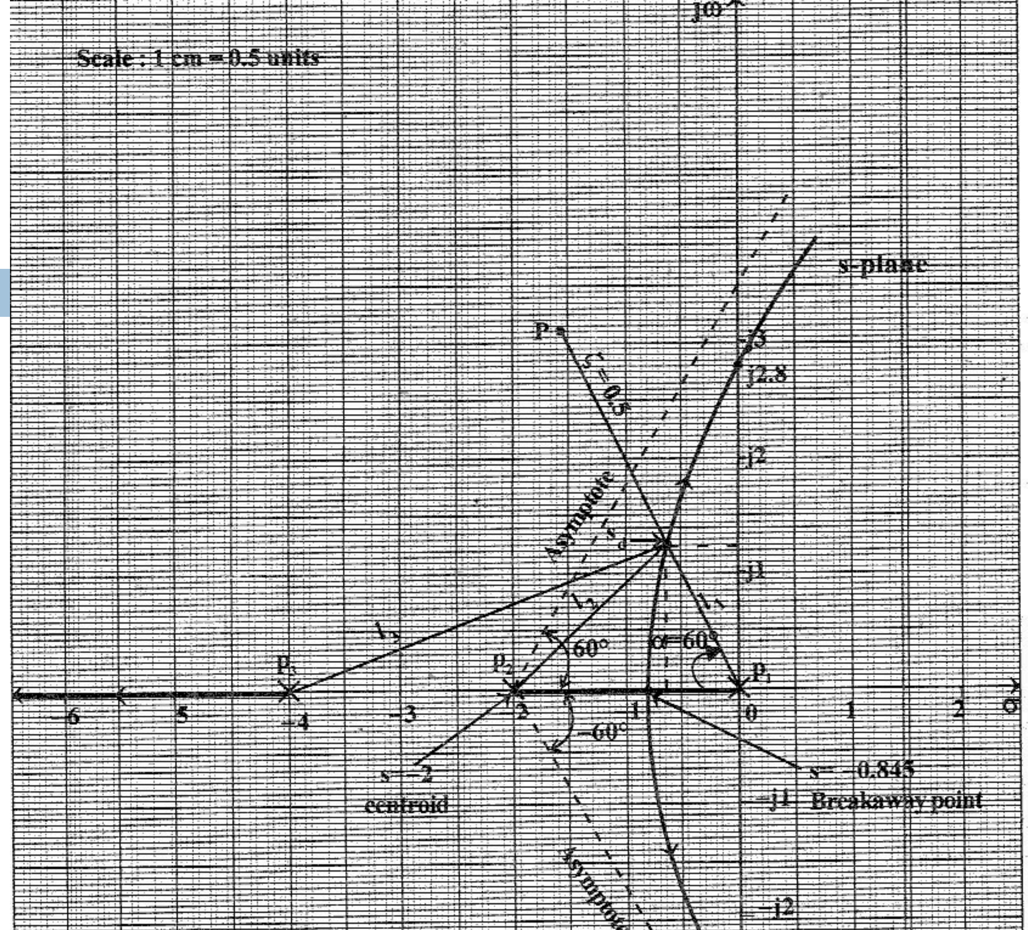
$$-j\omega^3 = -j8\omega$$

$$\omega^2 = 8 \quad \Rightarrow \quad \omega = \pm\sqrt{8} = \pm 2.8$$

Equating real part to zero

$$-6\omega^2 + K = 0$$

$$K = 6\omega^2 = 6 \times 8 = 48$$



$\alpha = \cos^{-1}\zeta = \cos^{-1} 0.5 = 60°$

$l_1 = 2.6\text{ cm} = 2.6 \times 0.5 = 1.3\text{ units}$

$l_2 = 3.5\text{ cm} = 3.5 \times 0.5 = 1.75\text{ units}$

$l_3 = 7\text{ cm} = 7 \times 0.5 = 3.5\text{ units}$

$K = l_1 \times l_2 \times l_3$

$= 1.3 \times 1.75 \times 3.5 = 7.96 \approx 8$

To find the value of K corresponding to ζ = 0.5

Given that $\zeta = 0.5$

Let $\alpha = \cos^{-1}\zeta = \cos^{-1} 0.5 = 60°$

Draw a line OP, such that the angle between line OP and negative real axis is 60° ($\alpha = 60°$) as shown in fig . The meeting point of the line OP and root locus gives the dominant pole, s_d.

Let K_{sd} be value of K corresponding to the p

$$K_{sd} = \frac{\text{Product of length of vector from all poles to the point, } s = s_d}{\text{Product of length of vector from all zeros to the point, } s = s_d}$$

$$= \frac{l_1 \times l_2 \times l_3}{1} = 1.3 \times 1.75 \times 3.5 = 7.96 \approx 8$$

Thank You